Progress in constructing epsilon form of differential equations for master integrals with Fuchsia

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Fuchsia is a program for reducing differential equations for master integrals to the epsilon form Henn '13:

- based on the Lee algorithm Lee '14
- open-source and free (no proprietary software dependencies)
- implemented in Python, SageMath, Maxima

The idea is to find a rational transformation in three reduction steps:

1. **Fuchsification** decrease Poincaré rank to 0 at all singular points (i.e. get rid of irregular singularities)
2. **Normalization**: balance eigenvalues to $n\epsilon$ form
3. **Factorization**: reduce to the epsilon form
4. Optimization for block-triangular systems (NEW!)

For more details see:

- Announcement at Loops&Legs ’16 Gituliar, Magerya [arXiv:1607.00759]
Problem

Calculate Feynman integrals of the form

\[ f_i(x, \epsilon) = \int \frac{d^d l_1 \ldots d^d l_n}{\text{loops}} \frac{d^d p_1 \delta(p_1^2) \ldots d^d p_m \delta(p_m^2)}{\text{legs}} \frac{1}{D_1^{n_1} \ldots D_k^{n_k}} \]

Numerical methods:
- sector decomposition (loops & legs)
- various subtraction schemes (legs)

Analytical methods:
- Integration-By-Parts (IBP) reduction to master integrals
  - Laporta algorithm: AIR, FIRE, Reduze
  - Symbolic reduction: LiteRed
- Feynman/Schwinger/Mellin-Barnes parametrization
- Ossola-Papadopoulos-Pittau (OPP) reduction for one-loop integrals
- Differential Equations for master integrals in the epsilon form
  - Lee algorithm: Fuchsia
Problem

Calculate Feynman integrals of the form

\[ f_i(x, \epsilon) = \int \underbrace{\prod_{l=1}^{n_{\text{loops}}} d^d l} \underbrace{\prod_{p=1}^{n_{\text{legs}}} d^d p_1 \delta(p_1^2) \ldots d^d p_m \delta(p_m^2)} \frac{1}{D_1^{n_1} \ldots D_k^{n_k}} \]

Let us consider a system of ODEs

\[ \frac{d\vec{f}}{dx} = \mathbb{A}(x, \epsilon) \vec{f}(x, \epsilon), \]

- \( \vec{f}(x, \epsilon) \) is a vector of unknown master integrals
- \( \mathbb{A}(x, \epsilon) \) is a singular matrix of rational functions

We are looking for the rational transformation \( \mathbb{T}(x, \epsilon) \) such that a new basis \( \vec{g}(x, \epsilon) \) is given by

\[ \vec{f}(x, \epsilon) = \mathbb{T}(x, \epsilon)\vec{g}(x, \epsilon) \]

with a new system in the \textbf{epsilon form} (easy to solve)

\[ \frac{d\vec{g}}{dx} = \epsilon \mathbb{A}(x) \vec{g}(x, \epsilon) \]
Example I: Splitting Functions at NLO in QCD

Splitting Functions for DGLAP evolution equations can be extracted from the $e^+e^- \rightarrow \gamma^* \rightarrow$ partons annihilation process:

\[
\begin{pmatrix}
\frac{(2\epsilon - 1)(2x - 1)}{x(1-x)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3\epsilon - 2}{x(1-x)} & \frac{1-3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 \\
\frac{x(1-x)}{(2\epsilon - 1)} & 0 & \frac{1-6\epsilon}{x+1} & 0 & 0 & 0 & 0 \\
\frac{\epsilon^2 x \times (1-x)(x+1)}{2(2\epsilon - 1)(1-x)} & \frac{(2\epsilon - 1)(3\epsilon - 1)}{x^2} & \frac{2(6\epsilon - 1)}{x(x+1)} & \frac{2\epsilon(x^2 + 3x - 2)}{(1-x)x(x+1)} & 0 & 0 & 0 \\
\frac{\epsilon^2 x^2 (x+1)}{2(x^2 + 4x + 1)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\epsilon^2(1-x)x^3(x+1)^3}{2(2\epsilon - 1)(1-x)^2x^3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4}{\epsilon^2(1-x)^3x^3(x+1)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Real-real contribution: initial form
Example 1: Splitting Functions at NLO in QCD

Splitting Functions for DGLAP evolution equations can be extracted from the $e^+ e^- \rightarrow \gamma^* \rightarrow$ partons annihilation process:

\[
\begin{pmatrix}
\frac{2}{1-x} - \frac{2}{x} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{19x} & -\frac{3}{x} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{19(1-x)} & \frac{2}{1-x} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{1-x} - \frac{2}{x} & 0 & 0 \\
\frac{3}{x} & -\frac{19}{x} & 0 & 0 & \frac{2}{1-x} + \frac{4}{x} & -\frac{1}{x} \\
\frac{6}{1+x} + \frac{6}{1-x} + \frac{16}{x} & -\frac{19}{x} & 0 & 0 & \frac{12}{1+x} + \frac{12}{1-x} + \frac{32}{x} & -\frac{2}{1+x} - \frac{8}{x} \\
\frac{19}{1-x} - \frac{19}{x} & \frac{4}{1+x} - \frac{4}{x} & \frac{76}{x} - \frac{76}{1-x} - \frac{16}{19} & 0 & \frac{4}{19(1-x)} + \frac{16}{19x} - \frac{8}{19(1-x)} & \frac{2}{1-x} - \frac{2}{x} \\
\frac{8}{19x} - \frac{6}{19(1+x)} & -\frac{6}{19x} & \frac{4}{1+x} - \frac{4}{x} & 0 & 0 & -\frac{2}{1+x} - \frac{2}{x}
\end{pmatrix}
\]

Now the system can be easily solved, hence giving initial master integrals by Fuchsia.

O. Gituliar [JHEP 1602 (2016) 017].
Example II: Two-loop planar pentabox

Papadopoulos, Tommasini, Wever [JHEP 1604 (2016) 078]

The latest optimization for block-triangular matrices allows Fuchsia to handle even more complicated problems:

Figure 3. The five-point Feynman diagrams, besides the pentabox itself in Figure 1, that are contained in the family $P_1$. All external momenta are incoming.
The resulting matrix for 74 master integrals

\[ M_{IJ} = N_{IJ}(\varepsilon) \left( \sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ:ik} \varepsilon^k}{(x - l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ:ik} \varepsilon^k x^j \right) \]

contains twenty letters \( l_i \) given by

\[
\begin{align*}
0, & \quad 1, \quad \frac{s_{45}}{s_{45} - s_{23}}, \quad \frac{s_{45}}{s_{12}}, \quad 1 - \frac{s_{34}}{s_{12}}, \quad 1 + \frac{s_{23}}{s_{12}}, \\
1 - \frac{s_{34} - s_{51}}{s_{12}}, & \quad \frac{s_{45} - s_{23}}{s_{12}}, \quad -\frac{s_{51}}{s_{12}}, \quad \frac{s_{45}}{-s_{23} + s_{45} + s_{51}}, \quad \frac{s_{45}}{s_{34} + s_{45}}, \\
\frac{s_{12}s_{23} - 2s_{12}s_{45} - s_{12}s_{51} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} \pm \sqrt{\Delta_1}}{2s_{12}(s_{23} - s_{45} - s_{51})}, & \quad \frac{s_{12}s_{23} - s_{12}s_{45} - s_{12}s_{51} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} \pm \sqrt{\Delta_2}}{2s_{12}(s_{23} - s_{45} - s_{51})}, \\
\frac{s_{12}s_{23} - s_{12}s_{51} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} \pm \sqrt{\Delta_1}}{2s_{12}(s_{23} + s_{34} - s_{51})}, & \quad \frac{s_{12}s_{45} \pm \sqrt{\Delta_3}}{s_{12}s_{34} + s_{12}s_{45}}, \quad \frac{s_{45}}{s_{12} + s_{23}},
\end{align*}
\]

where

\[
\begin{align*}
\Delta_1 &= (s_{12}(s_{51} - s_{23}) + s_{23}s_{34} + s_{45}(s_{51} - s_{34}))^2 + 4s_{12}s_{45}s_{51}(s_{23} + s_{34} - s_{51}) \\
\Delta_2 &= (s_{12}(-s_{23} + s_{45} + s_{51}) + s_{23}s_{34} + s_{45}(s_{51} - s_{34}))^2 - 4s_{12}s_{45}s_{51}(-s_{23} + s_{45} + s_{51}) \\
\Delta_3 &= -(s_{12}s_{34}s_{45}(s_{12} - s_{34} - s_{45}))
\end{align*}
\]
Example II: Two-loop planar pentabox

Initial matrix:

- optimizations for block-triangular shape
- 1.5 hours at i5 CPU
- using highly-optimized Maple routines
Fuchsia — a tool for reducing differential equations for master integrals to the epsilon form Henn '13:

- complete implementation of Lee '14 algorithm
- open-source and free (Python, SageMath, Maxima)
- powerful, e.g., two-loop planar pentabox
- [http://github.com/gituliar/fuchsia](http://github.com/gituliar/fuchsia)
- [http://gituliar.net/fuchsia/fuchsia.pdf](http://gituliar.net/fuchsia/fuchsia.pdf)

Prospects

- completely eliminate Maple dependencies
- more parameters (not only $x, \epsilon$)

Questions?